# Reasoning about Rough Sets Using Three Logical Values

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**Abstract.** The paper presents a logic for reasoning about covering-based rough sets using three logical values: the value **t** corresponding to the positive region of a set, the value **f** — to the negative region, and the undefined value **u** — to the boundary region of that set. Atomic formulas of the logic represent membership of objects of the universe in rough sets, and complex formulas are built out of the atomic ones using three-valued Kleene connectives. In the paper we provide a strongly sound and complete Gentzen-style sequent calculus for the logic.

## 1 Introduction

Rough sets, developed by Pawlak in the early 1980s [18, 19], represent a simple and yet very powerful notion designed to model vague or imprecise information. Unlike Zadeh's fuzzy sets, they are not based on any numerical measure of the degree of membership of an object in an imprecisely defined set. Instead, they employ a much more universal and versatile idea of an indiscernibility relation, which groups together objects having the same properties from the viewpoint of a certain application into disjoint equivalence classes.

This concept has proved to be extremely useful in practice. Since their introduction in the early 1980s, rough sets have found numerous applications in areas like control of manufacturing processes [14], development of decisions tables [20], data mining [14], data analysis [21], knowledge discovery [15], and so on. They have also been the subject of an impressive body of research. Though it focused mainly on algebraic properties of rough sets, a number of logicians have also explored this area, presenting and studying various brands of logics connected with rough sets [6, 5, 7, 8, 12, 16, 17, 10, 9, 23, 24].

Over the years, the original notion of rough sets has been generalized by replacing the indiscernibility relation (representing a partition of the universe of objects) underlying Pawlak's approach with other, less restrictive constructs. The broadest generalization are covering-based rough sets [27, 22], defined based on an arbitrary covering of the universe of objects. By now, this notion has also been examined in many papers (see e.g. [25, 26, 29]), with the main focus again on the algebraic properties of such generalized rough sets.

In this paper we explore the logical aspects of covering-based rough sets, employing for that purpose a three-valued logic. The motivation for using three logical values stems from the fact that, like in case of Pawlak's rough sets, a covering-based approximation space defines three regions of any set $X$ of objects:

– positive region of $X$, containing all objects of the universe which *certainly belong to* $X$ in the light of the information provided by the covering;
– negative region of $X$, containing all objects which *certainly do not belong to* $X$;
– boundary of $X$, containing all objects which *cannot be said for sure to either belong or not to belong to* $X$.

Hence the most natural idea for reasoning about membership of objects in covering-based rough sets is to use a three-valued logic, with the following values:

– **t** — meaning "certainly belongs", and assigned to objects in the positive region of a given set;
– **f** — meaning "certainly does not belong" and assigned to objects in the negative region of the set; and
– **u** — meaning "not known to either belong or not", and assigned to the boundary of the set.

Such an idea was first exploited in [3] for the original rough sets based on an equivalence relation on the universe of objects. However, the logic developed there was just a simple propositional logic generated by the three-valued non-deterministic matrix (see [4, 2]), shortly: Nmatrix, which did reflected only some properties of set-theoretic operations on rough sets.

Then next attempt was made in [13] for covering-based rough sets. There the semantics of the logics was based on the natural frameworks for such sets, i.e. covering-based approximations spaces. Atomic formulas of the logic represented either membership of objects of the universe in rough sets or the *subordination relation*[3] between objects generated by the covering underlying the approximation space, and complex formulas were formed out of the atomic ones using three-valued Kleene connectives. A Gentzen sequent calculus for the logic was presented, and its strong soundness was proved. However, strong completeness was only proved for the subset of the language containing formulas without set-theoretic operations on rough sets.

In this paper we improve on the results of [13] by providing a strongly sound and complete Gentzen calculus for the logic of rough sets defined as in [13], but without the subordination relation.

The paper is organized as follows. Section 2 presents the fundamentals of covering-based rough sets. Section 3 defines the syntax and semantics of the logic $\mathcal{L}_{RS}$ examined in the paper, including satisfaction and consequence relations

[3] Given a covering $\mathcal{C}$ of a universe $U$, the subordination relation generated by $\mathcal{C}$ is the binary relation $\prec_{\mathcal{C}}$ on $U$ such that $x \prec_{\mathcal{C}} y \Leftrightarrow (\forall C \in \mathcal{C})(y \in C \rightarrow x \in C)$. The relation $\prec_{\mathcal{C}}$ is reflexive and symmetric, and it is an equivalence relation iff $\mathcal{C}$ is a partition, i.e. in case of the original Pawlak's rough sets.

for formulas and sequents, Section 4 presents a strongly sound and complete Gentzen sequent calculus for $\mathcal{L}_{RS}$, and finally Section 5 presents the conclusions and outlines future work.

## 2 Covering-based rough sets

In what follows, for any set $X$, by $\mathcal{P}(X)$ we denote the powerset of $X$, i.e. the set of all subsets of $X$, and by $\mathcal{P}^+(X)$ — the set of all nonempty subsets of $X$.

**Definition 1.** By a *covering-based approximation space*, or shortly *approximation space*, we mean any ordered pair $\mathcal{A} = (U, \mathcal{C})$, where $U$ is a non-empty universe of *objects*, and $\mathcal{C} \subseteq \mathcal{P}^+(U)$ is a covering of $U$, i.e. $\bigcup\{C \mid C \in \mathcal{C}\} = \mathcal{U}$.

**Definition 2.** For any approximation space $\mathcal{A} = (U, \mathcal{C})$:

- The *lower approximation* of a set $X \subseteq U$ with respect to the covering $\mathcal{C}$ is

$$L_{\mathcal{C}}(X) = \{x \in U \mid \forall C \in \mathcal{C}(x \in C \Rightarrow C \subseteq X)\}$$

- The *upper approximation* of a set $X \subseteq U$ with respect to the covering $\mathcal{C}$ is

$$H_{\mathcal{C}}(X) = \bigcup\{C \in \mathcal{C} \mid C \cap X \neq \emptyset\}$$

In view of the above definition, one can say that, given the approximate knowledge about objects available in the approximation space $\mathcal{A}$:

- $L_{\mathcal{C}}(X)$ is the set of all the objects in $U$ which *certainly* belong to $X$;
- $H_{\mathcal{C}}(X)$ is the set of all the objects in $U$ which *might* belong to $X$;

The above operations have the same basic properties as in case of Pawlak's rough sets based on a partition of the universe, i.e. for any $X, Y \subseteq U$, we have:

$$L_{\mathcal{C}}(X) \subseteq X \subseteq H_{\mathcal{C}}(X)$$

$$\begin{array}{ll} H_{\mathcal{C}}(X \cup Y) = H_{\mathcal{C}}X \cup H_{\mathcal{C}}Y & L_{\mathcal{C}}(X \cup Y) \supseteq L_{\mathcal{C}}X \cup L_{\mathcal{C}}Y \\ L_{\mathcal{C}}(X \cap Y) = L_{\mathcal{C}}X \cap L_{\mathcal{C}}Y & H_{\mathcal{C}}(X \cap Y) \subseteq H_{\mathcal{C}}X \cap H_{\mathcal{C}}Y \\ L_{\mathcal{C}}(-X) = -H_{\mathcal{C}}X & H_{\mathcal{C}}(-X) = -L_{\mathcal{C}}X \end{array} \tag{1}$$

where none of the inequalities in (1) can be replaced by the equality.

Following the example of Pawlak's rough sets, with any subset of a universe $U$ of an approximation space we can associate three regions of that universe: positive, negative and boundary, representing three basic statuses of membership of an object of the universe $U$ in $X$:

**Definition 3.** Let $\mathcal{A} = (U, \mathcal{C})$ be an approximation space, and let $X \subseteq U$. Then:

- The *positive region* of $X$ in the space $\mathcal{A}$ with respect to the covering $\mathcal{C}$ is

$$POS_{\mathcal{C}}(X) = L_{\mathcal{C}}(X)$$

– The *negative region* of $X$ in the space $\mathcal{A}$ with respect to the covering $\mathcal{C}$ is

$$NEG_{\mathcal{C}}(X) = L_{\mathcal{C}}(U - X)$$

– The *boundary region* of $X$ in the space $\mathcal{A}$ with respect to the covering $\mathcal{C}$ is

$$BND_{\mathcal{C}}(X) = U - (POS_{\mathcal{C}}(X) \cup NEG_{\mathcal{C}}(X))$$

**Corollary 1.** *For any approximation space $\mathcal{A} = (U, \mathcal{C})$ and any $X \subseteq U$:*

$$\begin{aligned} &POS_{\mathcal{C}}(X) = \{x \in U \mid \forall C \in \mathcal{C}(x \in C \Rightarrow C \subseteq X)\} \\ &NEG_{\mathcal{C}}(X) = \{x \in U \mid \forall C \in \mathcal{C}(x \in C \Rightarrow C \subseteq U - X)\} \\ &BND_{\mathcal{C}}(X) = \{x \in U \mid \exists C \in \mathcal{C}(x \in C \wedge C \cap X \neq \emptyset \wedge C \cap (U - X) \neq \emptyset\} \end{aligned} \tag{2}$$

The regions defined as above are obviously disjoint. Moreover, we can say that, according to the approximate knowledge of the properties of objects in $U$ provided by the covering $\mathcal{C}$:

– The elements of $POS_R(X)$ *certainly belong* to $X$;
– The elements of $NEG_R(X)$ *certainly do not belong* to $X$;
– *We cannot tell if* the elements of $BND_R(X)$ *belong* to $X$ or not.

As a result, the most natural solution choice of a logic for reasoning about covering-based rough set is — exactly like in case of Pawlak's rough sets — to base it on three logical values: **t** — true, **f** – false, **u** — unknown, corresponding to the positive, negative and the boundary region of a set, respectively.

## 3 Syntax and semantics of the language $L_{RS}$

Now we shall define the language $L_{RS}$ of the three-valued logic for reasoning about covering-based rough sets described in the introduction. Formulas of $L_{RS}$ will contain expressions representing sets of objects (built from set variables and set constants using set-theoretic operators), variables representing objects, the symbol $\hat{\in}$ of a three-valued binary predicate representing membership of an object in a rough set, and the logical connectives $\neg, \wedge, \vee$ which will be interpreted as 3-valued Kleene connectives.

### 3.1 Syntax of $L_{RS}$

**Definition 4.** Assume that:

– $OV$ is a non-empty denumerable set of *object variables*;
– $SV$ is a non-empty denumerable set of *set variables*,
– **0** and **1** are *set constants*

The syntax of the language $L_{RS}$ is defined as follows:

1. The set $SE$ of *set expressions* of $L_{RS}$ is the least set containing $SV \cup \{\mathbf{0}, \mathbf{1}\}$, and closed under the set-theoretic operators $-, \cup, \cap$;
2. The set of *atomic formulas* of $L_{RS}$ is $\mathcal{A}_{RS} = \{x \hat{\in} A \mid x \in OV, A \in SE\}$;
3. The set $\mathcal{F}_{RS}$ of *formulas* of $L_{RS}$ is the least set $\mathcal{F}$ containing $\mathcal{A}_{RS}$ and closed under the connectives $\neg, \vee, \wedge$.

### 3.2 Semantic frameworks for $L_{RS}$ and interpretation of formulas

The semantics of $L_{RS}$ is based on interpreting the formulas of $L_{RS}$ in semantic frameworks for that language, built on covering-based approximation spaces and including valuations of set variables and object variables.

**Definition 5.** A *semantic framework*, or shortly *framework*, for $L_{RS}$ is an ordered triple $\mathcal{R} = (\mathcal{A}, v, w)$, where:

- $\mathcal{A} = (U, \mathcal{C})$ is a covering-based approximation space;
- $v : OV \to U$ is a valuation of object variables;
- $w : SV \to \mathcal{P}(U)$ is a valuation of set variables and constants such that $w(\mathbf{0}) = \emptyset, w(\mathbf{1}) = U$.

For any valuation $w : SV \to \mathcal{P}(U)$, by $w^*$ we shall denote the extension of $w$ to $SE$ obtained by interpreting $-, \cup, \cap$ as the set-theoretic operations of complement, union and intersection. In other words:

$$w^*(X) = w(X) \text{ for any } X \in SV, w^*(-A) = U - w(A) \text{ for any } A \in SE, \text{ and}$$

$$w^*(A \cup B) = w^*(A) \cup w^*(B), w^*(A \cap B) = w^*(A) \cap w^*(B) \text{ for any } A, B \in SE$$

**Definition 6.** An *interpretation* of $L_{RS}$ in a framework $\mathcal{R} = (\mathcal{A}, v, w)$, where $\mathcal{A} = (U, \mathcal{C})$, is a mapping $I_{\mathcal{R}} : \mathcal{F}_{RS} \to \{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$ defined as follows:

1. For any $x, y \in OV$ and any $A \in SE$,
$$I_{\mathcal{R}}(x \,\hat{\in} A) = \begin{cases} \mathbf{t} \text{ if } v(x) \in Pos_{\mathcal{C}}(w^*(A)) \\ \mathbf{f} \text{ if } v(x) \in Neg_{\mathcal{C}}(w^*(A)) \\ \mathbf{u} \text{ if } v(x) \in Bnd_{\mathcal{C}}(w^*(A)) \end{cases} .$$
2. For any $\varphi, \psi \in \mathcal{F}$,
   - $$I_{\mathcal{R}}(\neg\varphi) = \begin{cases} \mathbf{t} \text{ if } I_{\mathcal{R}}(\varphi) = \mathbf{f} \\ \mathbf{f} \text{ if } I_{\mathcal{R}}(\varphi) = \mathbf{t} \\ \mathbf{u} \text{ if } I_{\mathcal{R}}(\varphi) = \mathbf{u} \end{cases}$$
   - $$I_{\mathcal{R}}(\varphi \vee \psi) = \begin{cases} \mathbf{t} \text{ if either } I_{\mathcal{R}}(\varphi) = \mathbf{t} \text{ or } I_{\mathcal{R}}(\psi) = \mathbf{t} \\ \mathbf{f} \text{ if } I_{\mathcal{R}}(\varphi) = \mathbf{f} \text{ and } I_{\mathcal{R}}(\psi) = \mathbf{f} \\ \mathbf{u} \text{ otherwise} \end{cases}$$
   - $$I_{\mathcal{R}}(\varphi \wedge \psi) = \begin{cases} \mathbf{t} \text{ if } I_{\mathcal{R}}(\varphi) = \mathbf{t} \text{ and } I_{\mathcal{R}}(\psi) = \mathbf{t} \\ \mathbf{f} \text{ if either } I_{\mathcal{R}}(\varphi) = \mathbf{f} \text{ or } I_{\mathcal{R}}(\psi) = \mathbf{f} \\ \mathbf{u} \text{ otherwise} \end{cases}$$

It can be easily seen that the interpretation $I_{\mathcal{R}}$ is a well-defined mapping of the set of formulas into $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$. Indeed, as the regions of a rough set are disjoint, Point 1 provides a well-defined interpretation of atomic formulas, and the clauses for $\neg, \vee, \wedge$ in Point 2 extend it uniquely to complex formulas. In the sequel we will drop the subscript $\mathcal{R}$ at $I_{\mathcal{R}}$ if $\mathcal{R}$ is arbitrary or understood.

### 3.3 Satisfaction and consequence relations for formulas and sequents

To complete the definition of the semantics of $L_{RS}$, we need to define the notions of satisfaction and the consequence relation. Since the proof system we are going to develop for $L_{RS}$ will be a sequent calculus, we will define both the notions for formulas as well as for sequents.

**Definition 7.**

- *By a* sequent *we mean a structure of the form $\Gamma \Rightarrow \Delta$, where $\Gamma$ and $\Delta$ are finite* sets *of formulas. The set of all sequents over the language $L_{RS}$ is denoted by $Seq_{RS}$.*
- *A sequent $\Sigma \in Seq_{RS}$ is called* atomic *if each formula in $\Sigma$ is atomic.*

Depending on the specific application of rough sets, we can choose either the strong version of the three-valued semantics of $L_{RS}$ — with $\mathbf{t}$ as the only designated value, or its weak version — with two designated values: $\mathbf{t}, \mathbf{u}$, which leads to a paraconsistent logic. In this paper, we choose the strong semantics like in [3], leaving the weak version for future work, Consequently, we adopt the following definitions of satisfaction and consequence:

**Definition 8.** *1. A formula $\varphi \in \mathcal{F}_{RS}$ is* satisfied *by an interpretation $I$ of $L_{RS}$, in symbols $I \models \varphi$, if $I(\varphi) = \mathbf{t}$.*

2. *A formula $\varphi \in \mathcal{F}_{RS}$ is* valid*, in symbols $\models_{RS} \varphi$, if $I \models \varphi$ for any interpretation $I$ of $L_{RS}$.*
3. *A set of formulas $T \subseteq \mathcal{F}_{RS}$ is* satisfied *by an interpretation $I$, in symbols $I \models T$, if $I \models \varphi$ for all $\varphi \in T$.*
4. *A sequent $\Sigma = (\Gamma \Rightarrow \Delta)$ is* satisfied *by an interpretation $I$, in symbols $I \models \Sigma$, iff either $I \models \varphi$ for some $\varphi \in \Delta$, or $I \not\models \psi$ for some $\psi \in \Gamma$.*
5. *A sequent $\Sigma = (\Gamma \Rightarrow \Delta)$ is* valid*, in symbols $\models_{RS} \Sigma$, if $I \models \Sigma$ for any interpretation $I$ of $L_{RS}$*
6. *The* formula consequence relation *in $L_{RS}$ is the relation $\vdash_{RS}$ on $\mathcal{P}(\mathcal{F}_{RS}) \times \mathcal{F}_{RS}$ such that, for every $T \subset \mathcal{F}_{RS}$ and every $\varphi \in \mathcal{F}_{RS}, T \vdash_{RS} \varphi$ if each interpretation $I$ of $L_{RS}$ which satisfies $T$ satisfies $\varphi$ too.*
7. *The* sequent consequence relation *in $L_{RS}$ is the relation $\vdash_{RS}$ on $\mathcal{P}(Seq_{RS}) \times Seq_{RS}$ such that, for every $Q \subseteq Seq_{RS}$, and every $\Sigma \in Seq_{RS}, Q \vdash_{RS} \Sigma$ iff, for any interpretation $I$ of $L_{RS}, I \models_{RS} Q$ implies $I \models_{RS} \Sigma$.*

Note that the use of the same symbol for the formula and sequent consequence relations will not lead to misunderstanding, for the meaning of the symbol will always be clear from the context.

## 4 Proof system for the logic $\mathcal{L}_{RS}$

Now we shall present a proof system for the logic $\mathcal{L}_{RS}$ with the language $L_{RS}$, corresponding to the consequence relation $\vdash_{RS}$ defined in the preceding section. The deduction formalism we use for $\mathcal{L}_{RS}$ is a Gentzen-style sequent calculus.

**Sequent calculus CRS**

**Axioms:** $(A1)\ \varphi \Rightarrow \varphi$ $\qquad (A2)\ x\ \hat{\in}\mathbf{0} \Rightarrow$ $\qquad (A3) \Rightarrow x\ \hat{\in}\mathbf{1}$

**Structural rules: weakening, cut**

**Boolean tautology rules:** for any $A, B \in SE$ such that $A \equiv B$

$$(taut-l)\ \frac{\Gamma, x\ \hat{\in}A \Rightarrow \Delta}{\Gamma, x\ \hat{\in}B \Rightarrow \Delta} \qquad (taut-r)\ \frac{\Gamma \Rightarrow \Delta, x\ \hat{\in}A}{\Gamma \Rightarrow \Delta, x\ \hat{\in}B}$$

**Intersection rules:**

$$(\cap \Rightarrow)\ \frac{\Gamma, x\ \hat{\in}B, x\ \hat{\in}C \Rightarrow \Delta}{\Gamma, x\ \hat{\in}B \cap C \Rightarrow \Delta} \qquad (\Rightarrow \cap)\ \frac{\Gamma \Rightarrow \Delta, x\ \hat{\in}B \quad \Gamma \Rightarrow \Delta, x\ \hat{\in}C}{\Gamma \Rightarrow \Delta, x\ \hat{\in}B \cap C}$$

**Inference rules for Kleene connectives:**

$$(\neg\ \hat{\in} \Rightarrow)\ \frac{\Gamma, x\ \hat{\in} - A \Rightarrow \Delta}{\Gamma, \neg(x\ \hat{\in}A) \Rightarrow \Delta} \qquad (\Rightarrow \neg\ \hat{\in})\ \frac{\Gamma \Rightarrow \Delta, x\ \hat{\in} - A}{\Gamma \Rightarrow \Delta, \neg(x\ \hat{\in}A)}$$

$$(\neg\neg \Rightarrow)\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta} \qquad (\Rightarrow \neg\neg)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$$

$$(\vee \Rightarrow)\ \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma, \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\neg\vee \Rightarrow)\ \frac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta} \qquad (\Rightarrow \neg\vee)\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$$

$$(\wedge \Rightarrow)\ \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\neg\wedge \Rightarrow)\ \frac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta} \qquad (\Rightarrow \neg\wedge)\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$$

In all axioms and inference rules, we assume that $x, y, z \in OV,\ A, B \in SE$.

Note that though the axiom $\varphi, \neg\varphi$ is missing in the formulation of CRS, it is in fact derivable in CRS. Indeed, at the atomic level, from A1 and rule $(\neg \in\Rightarrow)$ we can deduce that (1) $\vdash_{CRS} x\ \hat{\in}A, \neg(x\ \hat{\in}A) \Rightarrow x\ \hat{\in}A, x\ \hat{\in} - A$. In turn, from A1 and rule $(\Rightarrow \cap)$ we can obtain (2) $\vdash_{CRS} x\ \hat{\in}A, x\ \hat{\in} - A \Rightarrow x \in (A \cap -A)$. Considering that $A \cap -A \equiv \mathbf{0}$, from rule $(taut-l)$ we can deduce that (3) $\vdash_{CRS} x \in (A \cap -A) \Rightarrow x\ \hat{\in}\mathbf{0}$. Applying cut, from 1), (2), (3) and Axiom A2 we conclude that $\vdash_{CRS} x\ \hat{\in}A, \neg(x\ \hat{\in}A) \Rightarrow$ , so $\varphi, \neg\varphi \Rightarrow$ holds for atomic $\varphi$. The fact that it holds for complex $\varphi$ too can be shown by structural induction using the inference rules for Kleene connectives .

**Lemma 1.**

1. *The axioms of the system CRS are valid.*
2. *For any inference rule $\rho$ of CRS and any framework $\mathcal{R}$ for $L_{RS}$, if the interpretation $I$ of $L_{RS}$ in $\mathcal{R}$ satisfies all the premises of $\rho$, then $I$ satisfies the conclusion of $\rho$ as well.*

Both parts can be easily verified based on the individual clauses of the definition of $I$ given in Definition 6.

Clearly, from the above Lemma we can immediately conclude that:

**Corollary 2.** *The inference rules of CRS are strongly sound, i.e. they preserve the validity of sequents.*

## 5 Strong soundness and completeness of the proof system

To prove strong completeness of CRS, we start with simple characterization of valid single-variable atomic sequents.

**Definition 9.** *For any $A, B \in SE$, we say that:*

1. *$A$ is* Boolean-equivalent *to $B$, and write $A \equiv B$, iff $A = B$ is a Boolean tautology;*
2. *$A$ is* Boolean-included *in $B$, and write $A \sqsubseteq B$, iff $A \cap B = A$ is a Boolean tautology, i.e. iff $A \cap B \equiv A$.*

**Proposition 1.** *A sequent $\Sigma = x \mathrel{\hat{\in}} A_1, \ldots, x \mathrel{\hat{\in}} A_k \Rightarrow x \mathrel{\hat{\in}} B_1, \ldots, x \mathrel{\hat{\in}} B_l$ is valid iff one of the following conditions is satisfied:*

(1) $A_1 \cap A_2 \cap \cdots \cap A_k \equiv \mathbf{0}$ (2) $B_i \equiv \mathbf{1}$ *for some* $i \leq l$

(3) $A_1 \cap A_2 \cap \cdots \cap A_k \sqsubseteq B_i$ *for some* $i \leq l$

*Proof.* The backward implication follows easily from the semantics of $\mathcal{L}_{RS}$. To prove the forward implication, we argue by contradiction. Assume that a sequent $\Sigma$ of the form given above is such that:

$(I)$ $A_1 \cap A_2 \cap \cdots \cap A_k \not\equiv \mathbf{0}$ $(II)$ $B_i \not\equiv \mathbf{1}$ for each $i \leq l$

$(III)$ $A_1 \cap A_2 \cap \cdots \cap A_k \not\sqsubseteq B_i$ for each $i \leq l$

Define

$$SV_0 = \{X \in SV \mid X \text{ occurs in } \Sigma\}$$

$$SE_0 = \{A \in SE \mid A \text{ contains only set variables in } SV_0\}$$

As $SV_0$ is finite, $SV_0 = \{X_1, X_2, \ldots, X_n\}$ for some $n$. The counter-model construction is based on the use of the full disjunctive normal form (DNF) of an expression in $SE_0$ with respect to $SV_0$. Such a DNF is of the form

$$\mathbf{X}^{\overline{\epsilon}} = X_1^{\epsilon_1} \cap X_2^{\epsilon_2} \cap \cdots \cap X_n^{\epsilon_n}$$

where $\bar{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{-1, 1\}^n$, and $X_j^1 = X_j, X_j^{-1} = -X_j$.

Let $A = A_1 \cap A_2 \cap \cdots \cap A_k$. Then $A \not\equiv \mathbf{0}$ by (I), so we have

$$\mathrm{DNF}(A) = \mathbf{X}^{\bar{\epsilon}_1} \cup \mathbf{X}^{\bar{\epsilon}_2} \cup \cdots \cup \mathbf{X}^{\bar{\epsilon}_p}$$

for some $p \geq 1, \bar{\epsilon}_1, \cdots, \bar{\epsilon}_p \in \{-1, 1\}^n$. Since $\mathrm{DNF}(E) \equiv E$ for any $E \in SE_0$, then by (III) we get $\mathrm{DNF}(A) \not\sqsubseteq \mathrm{DNF}(B_i)$ for each $i \leq l$. Hence for each $i \leq l$ there is a $j_i \leq p$ such that $\mathbf{X}^{\bar{\epsilon}_{j_i}}$ does not occur in $\mathrm{DNF}(B_i)$.

Let us assign a unique symbol $a^{\bar{\epsilon}} \notin OV \cup SV$ to any $\bar{\epsilon} \in \{-1, 1\}^n$. As the universe of our counter-model $\mathcal{R}$ we take $U = \{x\} \cup \{a^{\bar{\epsilon}} \mid \bar{\epsilon} \in \{-1, 1\}^n\}$, and as the covering underlying the approximation space — $\mathcal{C} = \{C(u) \mid u \in U\}$, where $C(u) = \{u\}$ for $u \neq x$, and $C(x) = \{x, a^{\bar{\epsilon}_1}, a^{\bar{\epsilon}_2}, \ldots, a^{\bar{\epsilon}_p}\}$. The valuation of variables $v$ is given by $v(y) = x$ for any $x \in OV$. Finally, to define the valuation of set variables, we first define $\xi(\mathbf{X}^{\bar{\epsilon}_1}) = \{x, a^{\bar{\epsilon}_1}\}$ and $\xi(\mathbf{X}^{\bar{\epsilon}}) = \{a^{\bar{\epsilon}}\}$ for any $\bar{\epsilon} \in \{-1, 1\}^n \setminus \{\bar{\epsilon}_1\}$. Then we put $w(X) = \emptyset$ for $X \in SV \setminus SV_0$, and define $w$ on $SV_0$ by taking

$$w(X_j) = \bigcup \{\xi(\mathbf{X}^{\bar{\epsilon}}) \mid \bar{\epsilon} \in \{-1, 1\}^n, \epsilon_j = 1\}$$

for $j = 1, 2, \ldots, n$ (recall that $SV_0 = \{X_1, X_2, \ldots, X_n\}$). It is easy to check that $\mathcal{R} = ((U, \mathcal{C}), v, w)$ is a well-defined semantic framework for $L_{RS}$, and

$$w^*(A_1 \cap A_2 \cap \cdots A_k) = w^*(X) = \{x, a^{\bar{\epsilon}_1}, a^{\bar{\epsilon}_2}, \ldots, a^{\bar{\epsilon}_p}\} = C(x) \tag{3}$$

However, as $w^*(A_1 \cap A_2 \cap \cdots \cap A_k) = \bigcap_{r=1}^{k} w^*(A_r) \subseteq w^*(A_j)$ for each $j \leq k$, (3) implies that $C(x) \subseteq w^*(A_j)$ for any $j \leq k$, Since $C(x)$ is the only set $C \in \mathcal{C}$ such that $x \in C$, then from Corollary 1 we obtain $x \in POS(w^*(A_j))$ and $I_{\mathcal{R}} \models x \mathrel{\hat{\in}} A_j$ for $j = 1, 2, \ldots, k$. On the other hand, as $\mathbf{X}^{\bar{\epsilon}_{j_i}}$ does not occur in $\mathrm{DNF}(B_i)$ for any $i \leq l$, then $a^{\bar{\epsilon}_{j_i}} \notin w^*(\mathrm{DNF}(B_i)) = w^*(B_i)$ for each $i \leq l$, which in view of $a^{\bar{\epsilon}_{j_i}} \in C(x)$ implies $C(x) \not\subseteq w^*(B_i)$ for each $i \leq l$. As a result, $I_{\mathcal{R}} \not\models x \mathrel{\hat{\in}} B_i$ for $i = 1, 2, \ldots, l$. Thus $I_R \not\models \Sigma$, which ends the proof.

Since $L_{RS}$ has no means for expressing relationships between object variables, Proposition 1 implies a similar result for multi-variable atomic sequents:

**Corollary 3.** *An atomic sequent $\Sigma \in Seq_{RS}$ is valid if and only if, for some object variable $x$ occurring in $\Sigma$, the sequent $\Sigma_x$ obtained from $\Sigma$ by deleting all formulas with variables different from $x$ satisfies one of the conditions of Proposition 1.*

The proof is analogous to that of Proposition 1, with the counter-model for a sequent $\Sigma$ which does not satisfy any of conditions (1),(2),(3) of that Proposition constructed by combining the individual countermodels for all single-variable subsequents of $\Sigma$, constructed exactly like in the proof of Proposition 1.

From the results of the preceding section, we can easily conclude that CRS is complete for atomic sequents:

**Proposition 2.** *If an atomic sequent $\Sigma \in Seq_{RS}$ is valid, then it is derivable in CRS, i.e. $\vdash_{CRS} \Sigma$.*

*Proof.* For any variable $x$ occurring in $\Sigma$, denote by $\Sigma_x$ the atomic sequent obtained out of $\Sigma$ by deleting all formulas with variables different from $x$. Since $\Sigma$ is valid, then, by Corollary 3, there exists an $x$ such that $\Sigma_x$ satisfies one of the conditions (1), (2), (3) of Proposition 1. Hence, assuming that $\Sigma_x = x \hat{\in} A_1, \ldots, x \hat{\in} A_k \Rightarrow x \hat{\in} B_1, \ldots, x \hat{\in} B_l$, we have

either (1) $A_1 \cap A_2 \cap \cdots \cap A_k \equiv \mathbf{0}$ or (2) $B_i \equiv \mathbf{1}$ for some $i \leq l$

or (3) $A_1 \cap A_2 \cap \cdots \cap A_k \sqsubseteq B_i$ for some $i \leq l$

If (1) holds, then from A1 and rule $(\Rightarrow \cap)$ applied $k-1$ times we can obtain (i) $\vdash_{CRS} x \hat{\in} A_1, \ldots, x \hat{\in} A_k \Rightarrow x \hat{\in} (A_1 \cap \cdots \cap A_k)$. Considering that $A_1 \cap \cdots \cap A_k \equiv \mathbf{0}$, from rule $(taut - l)$ and Axioms A1, A2 we get (ii) $\vdash_{CRS} x \hat{\in} (A_1 \cap \cdots \cap A_k) \Rightarrow$ . Applying cut to (i) and (ii), we obtain $\vdash_{CRS} x \hat{\in} A_1, \ldots, x \hat{\in} A_k \Rightarrow$ , whence $\vdash_{CRS} \Sigma_x$ by weakening.

If (2) holds, then from Axiom A3 and rule $(taut - r)$ we get $\vdash_{CRS} \Rightarrow B_i$, whence $\vdash_{CRS} \Sigma_x$ by weakening.

Finally, assume that (3) holds. By what was shown for (1), we have (i) $\vdash_{CRS} x \hat{\in} A_1, \ldots, x \hat{\in} A_k \Rightarrow x \hat{\in} (A_1 \cap \cdots \cap A_k)$. For simplicity, denote $A = A_1 \cap \cdots \cap A_k$. Then $A \sqsubseteq B_i$, implying $A \cap B_i \equiv A$ by Definition 9, whence from Axiom A1 and rule $(taut - l)$ we get (ii) $\vdash_{CRS} x \hat{\in} A \Rightarrow x \hat{\in} A \cap B_i$. In turn, by A1 and rule $(\cap \Rightarrow)$ we have (iii) $\vdash_{CRS} x \hat{\in} A \cap B_i \Rightarrow x \hat{\in} B_i$. Applying cut twice to (i), (ii) and (iii), we obtain $\vdash_{CRS} x \hat{\in} A_1, \ldots, x \hat{\in} A_k \Rightarrow x \hat{\in} B_i$, which yields $\vdash_{CRS} \Sigma_x$ by weakening.

Thus $\vdash_{CRS} \Sigma_x$ in all three cases. As $\Sigma_x \subset \Sigma$ in the standard sense of sequent inclusion, this implies $\vdash_{CRS} \Sigma$ by weakening.

Proposition 2 is the cornerstone for proving the strong completeness theorem for the logic $\mathcal{L}_{RS}$:

**Theorem 1.** *The calculus CRS is finitely strongly sound and complete for $\vdash_{RS}$, i.e., for any finite set of sequents $S \subseteq Seq_{RS}$ and any sequent $\Sigma \in Seq_{RS}$, $S \vdash_{RS} \Sigma$ iff $S \vdash_{CRS} \Sigma$.*

*Proof.* (Sketch) As the backward implication (soundness) follows from Lemma 1 and Corollary 2, it suffices to prove the forward implication (completeness). The proof is by counter-model construction based on Proposition 2 and the maximum saturated sequent construction used e.g. in [1].

We argue by contradiction. Suppose that for a finite set of sequents $S$ and a sequent $\Sigma = \Gamma \Rightarrow \Delta$ we have $S \vdash_{RS} \Sigma$, but $\Sigma$ is not derivable from $S$ in CRS. We shall construct a counter-model $I$ such that $I \models S$ but $I \not\models \Sigma$.

Denote by $F(S)$ the set of all formulae belonging to at least one of the sides in some sequent in $S$, and let $SV^*$ be the set of all set variables which occur either in some $\varphi \in F(S)$ or in $\Sigma$. Since $S$ is finite, so are $F(S)$ and $SV^*$. Using the method shown in in [1], we can construct a sequent $\Gamma' \subseteq \Delta'$ such that

(i) $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$
(ii) $F(S) \subseteq \Gamma' \cup \Delta'$.

(iii) $\Gamma' \Rightarrow \Delta'$ is not derivable from $S$ in CRS.

The construction is carried out by starting with $\Sigma$, and then adding consecutively linearly ordered formulas in $F(S)$ to either the left- or the right-hand side of the sequent constructed up to that time, depending on which option results in a sequent still not derivable from $S$ in CRS. Such a construction is possible because if $S \not\vdash_{CRS} (\Gamma_i \Rightarrow \Delta_i)$, then, for any $\varphi \in F(S)$, we cannot have both $S \vdash_{CRS} (\Gamma_i, \varphi \Rightarrow \Delta_i)$ and $S \vdash_{CRS} (\Gamma_i \Rightarrow \Delta_i, \varphi)$, since this would imply $S \vdash_{CRS} (\Gamma_i \Rightarrow \Delta_i)$ by cut.

Call a sequent *saturated* if it is closed under the inference rules in CRS applied backwards, whereby we assume that closure under the Boolean tautology rules $(taut - l), (taut - r)$ is limited only to premises with the set expression $A$ in a full disjunctive normal form with respect to the set $SV^*$. By way of example, a sequent $\Gamma'' \Rightarrow \Delta''$ is closed under rule $(\vee \Rightarrow)$ applied backwards iff $\varphi \vee \psi \subseteq \Gamma''$ implies either $\varphi \in \Gamma''$ or $\psi \in \Gamma''$.

Let $\Gamma^* \Rightarrow \Delta^*$ be the extension of $\Gamma' \Rightarrow \Delta'$ to a saturated sequent which is not derivable from $F(S)$ in $CRS$ (is is easy to see that such a sequent exists; note that the restriction on the closure under tautology rules ensures that the closure adds only a finite number of formulas to $\Gamma' \Rightarrow \Delta'$.

Then we can easily see that:

(1) $\Gamma \subseteq \Gamma^*, \Delta \subseteq \Delta^*$;
(2) $F(S) \subseteq \Gamma^* \cup \Delta^*$;
(3) $\Gamma^* \Rightarrow \Delta^*$ is saturated and it is not derivable from $S$ in CRS.

Now let $\Sigma_a = \Gamma_a \Rightarrow \Delta_a$ be a subsequent of $\Gamma^* \Rightarrow \Delta^*$ consisting of all atomic formulas in $\Gamma^* \Rightarrow \Delta^*$. Then by (3) $S \not\vdash_{CRS} \Sigma_a$, and hence also $\not\vdash_{CRS} \Sigma_a$. As $\Sigma_a$ is atomic, by Proposition 2, this implies that $\Sigma_a$ is not valid. Accordingly, there exists a framework $\mathcal{R}$ for $L_{RS}$ and an interpretation $I$ of $L_{RS}$ in $\mathcal{R}$ such that $I \not\models \Sigma_a$. We shall prove that $I$ is the desired counter-model for the original sequent $\Sigma$ too, i.e. that:

$$(\mathbf{A})\ I \not\models (\Gamma \Rightarrow \Delta) \qquad (\mathbf{B})\ I \models \Sigma \text{ for each } \Sigma \in S$$

Let us start with (A). As $\Gamma \subseteq \Gamma^*, \Delta \subseteq \Delta^*$, then in order to prove (A) it suffices to show that $I \not\models (\Gamma^* \Rightarrow \Delta^*)$. Since the only designated value in the semantics of $L_{RS}$ is $\mathbf{t}$ and $I(\varphi) \in \{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$ for any formula $\varphi \in F_{RS}$, this means we have to prove that:

$$I(\gamma) = \mathbf{t} \text{ for any } \gamma \in \Gamma^* \qquad I(\delta) \in \{\mathbf{f}, \mathbf{u}\} \text{ for any } \delta \in \Delta^* \tag{4}$$

As $I \not\models \Sigma_a$, then (4) holds for all atomic formulas $\gamma \in \Gamma^*, \delta \in \Delta^*$. To show that it holds for complex formulas too, we prove that, for any complex formula $\varphi$, the following is true:

$$(\mathbf{A1}) \qquad I(\varphi) = \begin{cases} \mathbf{t} \text{ if } \varphi \in \Gamma^* \\ \mathbf{f} \text{ if } \neg\varphi \in \Gamma^* \end{cases} \qquad (\mathbf{A2}) \qquad I(\varphi) \in \begin{cases} \{\mathbf{f}, \mathbf{u}\} \text{ if } \varphi \in \Delta^* \\ \{\mathbf{t}, \mathbf{u}\} \text{ if } \neg\varphi \in \Delta^* \end{cases}$$

The proof is by induction on the complexity of $\varphi$, and (A1) and (A2) are proved simultaneously, making use of the fact that $\Sigma^*$ as a saturated sequent is closed under all rules in CRS applied backwards.

To illustrate the method used, consider first the case of $\xi = \neg(x \,\hat{\in} A)$.

If $\xi \in \Gamma^*$, then $x \,\hat{\in} - A$ is also in $\Gamma^*$, since $\Sigma^*$ is closed under rule $(\neg \,\hat{\in} \Rightarrow)$ applied backwards. As (4) holds for all atomic formulas and $x \,\hat{\in} - A$ is atomic, this yields $I(x \,\hat{\in} - A) = \mathbf{t}$. However, from Definition 6 and Corollary 1 we can easily conclude that

$$I(x \,\hat{\in} A) = \begin{cases} \mathbf{t} & \text{iff } I(x \,\hat{\in} - A) = \mathbf{f} \\ \mathbf{f} & \text{iff} I(x \,\hat{\in} - A) = \mathbf{t} \\ \mathbf{u} & \text{iff} I(x \,\hat{\in} - A) = \mathbf{u} \end{cases} \tag{5}$$

which implies $I(\xi) = I(\neg(x \,\hat{\in} A)) = \mathbf{t}$ by Definition 6.

In turn, if $\xi \in \Delta^*$, then $x \,\hat{\in} - A$ is also in $\Delta^*$, since $\Sigma^*$ is closed under rule $(\Rightarrow \neg \,\hat{\in})$ applied backward. As (4) holds for $x \,\hat{\in} - A$, then $I(x \,\hat{\in} - A) \in \{\mathbf{f}, \mathbf{u}\}$, whence in view of (5) we get $I(x \,\hat{\in} A) \in \{\mathbf{t}, \mathbf{u}\}$. In consequence, $I(\xi) = I(\neg(x \,\hat{\in} A)) \in \{\mathbf{f}, \mathbf{u}\}$ by Definition 6. Thus (A1) and (A2) hold for $\xi$

As another example, assume that (A1), (A2) hold for $\varphi, \psi$, and that $\xi = \varphi \vee \psi$. If $\xi \in \Gamma^*$, then either $\varphi \in \Gamma^*$ or $\psi \in \Gamma^*$, since $\Sigma^*$ is closed under rule $(\vee \Rightarrow)$ applied backwards. As a result, by the inductive assumption on $\varphi, \psi$ we have either $I(\varphi) = \mathbf{t}$ or $I(\psi) = \mathbf{t}$, and consequently $I(\xi) = \mathbf{t}$ by Definition 6. In turn, if $\xi \in \Delta^*$, then $\varphi, \psi \in \Delta^*$, and $I(\varphi), I(\psi) \in \{\mathbf{f}, \mathbf{u}\}$ by the inductive assumption, whence $I(\xi) \in \{\mathbf{f}, \mathbf{u}\}$ by Definition 6, too. As a result, (A1) and (A2) hold for $\xi$ too.

The proof of other cases is similar, and is left to the reader.

It remains to prove (B), i.e., to show that $I \models \Sigma_0$ for each $\Sigma_0 \in S$. So let $\Sigma_0 \in S$. Then $\Sigma_0 = \varphi_1, \ldots, \varphi_k \Rightarrow \psi_1, \ldots, \psi_l$ for some integers $k, l$ and formulas $\varphi_i, \psi_j, i = 1, \ldots, k, j = 1, \ldots, l$. Clearly, we cannot have both $\{\varphi_1, \ldots, \varphi_k\} \subseteq \Gamma^*$ and $\{\psi_1, \ldots, \psi_l\} \subseteq \Delta^*$, for then $\Gamma^* \Rightarrow \Delta^*$ would be derivable from $\Sigma_0$, and hence from $S$, by weakening. Since $F(S) \subseteq \Gamma^* \cup \Delta^*$, this implies that either $\varphi_i \in \Delta^*$ for some $i$, or $\psi_j \in \Gamma^*$ for some $j$. Hence by (A1) and (A2), which we have already proved, we have either $I \not\models \varphi_i$ for some $i$, or $I \models \psi_j$ for some $j$, which implies that $I \models \Sigma$.

## 6 Conclusions and future work

The crucial feature of three-valued logic presented in the paper is a complete mechanism for reasoning about atomic formulas representing three-valued, rough membership of the objects of the universe in rough sets. The three values taken by the rough membership relation correspond to “crisp” membership of objects in the three basic regions of a rough set: the positive, negative and boundary one. The strong version of semantics with the single designated value $\mathbf{t}$ adopted in the paper amounts to identifying membership of an object $x$ in a rough set $A$ with its belonging to the positive region of $A$. However, in many applications it is advisable to identify the above membership with $x$ belonging to either the positive region or the boundary region of $A$ — which corresponds to taking also **u** as a designated value. The latter option, which leads to paraconsistent logic, will be the subject of further work.

The use of connectives to form complex formulas enhances the expressive power of the language, but the Kleene 3-valued connectives used here are just one possible choice. Other interesting option, to be explored in the future, are the Łukasiewicz 3-valued connectives (including implication), and the non-deterministic connectives observing the rough set Nmatrix considered in [3]. Exploring these choices is another direction for future work. A still another is to consider a richer language which allows for expressing relationships between objects — and here the most immediate future task will be extending the results of this paper to a language featuring the subordination relation of [13].

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